A Polynomial Representation of sl2R

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1 One Degree of Freedom

With a single coordinate q, with associated momentum p, Phase space can be modeled by $\mathbb{R}^2$. Let A and B be two real functions of $\mathbb{R}^2$. The Poisson bracket on $\mathbb{R}^2$ is given by

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}.$$

This bracket is a Lie bracket and hence $\mathbb{R}^2$ is a Lie algebra.

The Lie algebra of analytic functions over $\mathbb{R}^2$ has an important subspace, the space of finite polynomials $\mathbb{R}[q, p]$. Because each monomial in this space is linearly independant[1], the bilinearity of the Poisson bracket implies that we need only consider Lie brackets of the form,

$$\{q^a p^b, q^c p^d\} = (ad - bc) q^{a+c-1} p^{b+d-1}.$$

Therefore, we see explicitly that $\mathbb{R}[q, p]$ is a Lie subalgebra.

The **degree** of a monomial is the sum of its exponents,

$$\deg q^a p^b = a + b.$$

We then see that the Poisson bracket of a monomial with degree m with a monomial of degree n is $m+n-2$. That is to say,

$$\deg\{q^a p^b, q^c p^d\} = a + b + c + d - 2. \tag{1}$$

Polynomials whose terms all share the same degree are said to be of **homogeneous degree**. In this language, $\mathbb{R}[q, p]$ can be written as the direct sum of subspaces of homogeneous degree,

$$\mathbb{R}[q, p] = \bigoplus_{n=1}^{\infty} P_n,$$

where the linear subspace of homogeneous polynomials of degree n is given by

$$P_n = \mathsf{span}\left\{ q^a p^b \;\middle|\; a + b = n \right\}.$$

[1]If this is not immediatley clear to you, verify this claim as an exercise.

2 A Polynomial Representation of $\mathfrak{sl}(2,\mathbb{R})$

Proposition 2.1. Under the Poisson bracket, P_2 is a subalgebra of $\mathbb{R}[q,p]$.

Proof. Let both $a+b$ and $c+d$ equal two. Then by (1), we see that $\deg\{q^ap^b, q^cp^d\}$ also equals two. $\square$

The space P_2 is three-dimensional, as there are precisely three quadratic monomials,

$$P_2 = \mathsf{span}\{q^2, p^2, qp\}.$$

Restricted to P_2, the Poisson bracket is nonsingular, as we can see by direct computation,

$$\{q^2, p^2\} = 4qp, \quad \{qp, q^2\} = -2q^2, \quad \{qp, p^2\} = 2p^2. \tag{2}$$

Proposition 2.2. P_2 is isomorphic as a Lie algebra to $\mathfrak{sl}(2,\mathbb{R})$.

Proof. We prove this by brute force. First recall that $\mathfrak{sl}(2,\mathbb{R})$ is a three-dimensional vector space of traceless matrices,

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is a Lie algebra under the usual commutator of matrices, which are given explicitly by

$$[\sigma, \sigma_\pm] = \pm 2\sigma_\pm, \quad [\sigma_+, \sigma_-] = \sigma.$$

Both Lie algebras P_2 and $\mathfrak{sl}(2,\mathbb{R})$ are three-dimensional. Hence if we can show that they have identical Lie brackets, we are done. To that end, define,

$$\sigma \leftarrow -qp, \quad \sigma_+ \leftarrow -\frac{1}{2}q^2, \quad \sigma_- \leftarrow \frac{1}{2}p^2.$$

We can then compute, using (2),

$$\{\sigma, \sigma_+\} = \frac{1}{2}\{qp, q^2\} = -q^2 = 2\left(-\frac{1}{2}q^2\right) = 2\sigma_+.$$

Also

$$\{\sigma, \sigma_-\} = \frac{1}{2}\{-qp, p^2\} = -p^2 = -2\left(\frac{1}{2}p^2\right) = -2\sigma_-.$$

Finally,

$$\{\sigma_+, \sigma_-\} = -\frac{1}{4}\{q^2, p^2\} = -qp = \sigma.$$

Hence all brackets agree and P_2 is therefore isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.

$\square$

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3 The Adjoint Representation

Recall that to each element g of a Lie algebra $\mathfrak{g}$, we can write the adjoint operator ad_g,

$$\mathsf{ad}_g : \mathfrak{g} \to \mathfrak{g},$$

where for each h in $\mathfrak{g}$,

$$\mathsf{ad}_g : h \mapsto [h, g].$$

Using the exponential map, we can promote the adjoint operator to a member of the Lie group[2]. Let θ be an constant parameter, this group member is given by

$$e^{\theta \mathsf{ad}_g} = \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n \mathsf{ad}_g^n.$$

Let us consider a specific example in the case of P_2. In particular, we've seen that the matrix $\sigma_- - \sigma_+$ gives us the antisymmetric matrix K that generates rotations,

$$K = \sigma_- - \sigma_+ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let us verify this remains true for the case of P_2. In this instance we have

$$K \leftarrow \frac{1}{2}\left(q^2 + p^2\right).$$

Let us compute the adjoint action of K on the coordinate q.

$$\mathsf{ad}_K \cdot q = \{q, K\} = \frac{1}{2}\{q, p^2\} = p.$$

Similarly we can compute the next few powers of ad_K,

$$\mathsf{ad}_K^2 \cdot q = \{p, K\} = \frac{1}{2}\{p, q^2\} = -q.$$

Again,

$$\mathsf{ad}_K^3 \cdot q = \{-q, K\} = -p.$$

Finally, we see that this pattern repeats, where

$$\mathsf{ad}_K^4 \cdot q = \{-p, K\} = -\frac{1}{2}\{p, q^2\} = q.$$

Iterating, we see that

$$\mathsf{ad}_K^{2n} \cdot q = (-1)^n q, \quad \mathsf{ad}_K^{2n+1} \cdot q = (-1)^n p.$$

Hence,

$$e^{\theta \mathsf{ad}_k} \cdot q = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \theta^{2n} q + \frac{(-1)^n}{(2n+1)!} \theta^{2n+1} p = \cos\theta q + \sin\theta p.$$

So K does indeed generate rotations in P_2, the polynomial representation of $\mathfrak{sl}(2, \mathbb{R})$.

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[2]This group member then acts on the Lie algebra.

Exercise 1. Repeat this analysis for the hypoerbolic rotations, $(\sigma_+ + \sigma_-)$, and the dilatatons, σ. What operator corresponds to the Nilpotent matrix $N(x)$?