The Pasayten Institute

The Lie Algebra sl2nR

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In this episode we describe the Lie algebra of matrices $\mathfrak{sp}(2n, \mathbb{R})$ that generate the linear symplectomorphisms, $\mathsf{Sp}(2n, \mathbb{R})$.

1 $\mathsf{SL}(2, \mathbb{R})$ as a manifold

Let's warm up with the case where $n = 1$.

Previously we have seen that the dimension of $\mathsf{SL}(2, \mathbb{R})$ is three. This is the dimension in a manifold sense, which is to say $\mathsf{SL}(2, \mathbb{R})$ represents a space that looks locally like $\mathbb{R}^3$, but globally has nontrivial structure[1].

The Lie algebra of generators, however, can be thought of as those matrices infinitestimally close to the identity. Geometrically speaking, some might call a coordinate chart near the identity, or, by extension, a *tangent space*. One way to model this idea is to use the exponential map.

Let us write a member M of $\mathsf{SL}(2, \mathbb{R})$ in terms of a matrix Q,

$$M = e^Q = \sum_{n=0}^{\infty} \frac{1}{n!} Q^n.$$

The condition that $\det M = 1$ translates into a condition on Q,

$$\det M = e^{\mathsf{Tr}\, Q},$$

so that $\det M = 1$ implies $\mathsf{Tr}\, Q$ vanishes[2].

Up to constant factors, there are precisely three two-by-two, traceless matrices,

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

These span the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ which generates $\mathsf{Sp}(2n, \mathbb{R})$.

[1] That is to say, a small enough neighborhood around a point will be indistguishable from a vector space. Curiously, the three-dimensional manifold $\mathsf{SL}(2, \mathbb{R})$ cannot even be properly immersed in $\mathbb{R}^3$, which means that any attempt to visualize what the group might look like would not fairly represent the group.

[2] You can derive the trace relation using Jacobi's formula for the derivative of a determinant.

1.1 The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$

Because there are only three linearly independent matrices, it is worthwhile to compute the explicit Lie brackets to determine the precise algebraic relationships between members of $\mathfrak{sl}(2, \mathbb{R})$.

Exercise 1. Verify by explicit computation that

$$[\sigma, \sigma_\pm] = \pm 2\sigma_\pm, \quad \text{and} \quad [\sigma_+, \sigma_-] = \sigma.$$

1.2 Examples

Let us revisit some familiar examples by showing how the Lie algebra generates the one-parameter subgroups of $\mathsf{SL}(2, \mathbb{R})$ we've already discussed.

Define the matrix K as

$$K = \sigma_- - \sigma_+ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 1.1. The rotation matrix $R(\theta)$ can be written as

$$R(\theta) = e^{\theta K}.$$

Proof. Let us expand the exponential,

$$e^{\theta K} = \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n K^n.$$

It helps to observe that

$$K^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathbb{I}.$$

Hence $K^3 = -K$. Hence $K^4 = \mathbb{I}$. This cycle then repeats. Notably, even powers of K are proportional to $\mathbb{I}$ and hence odd powers are proportional to K. In particular,

$$K^{2n} = (-1)^n \mathbb{I}, \quad K^{2n+1} = (-1)^n K.$$

Plugging these results into the exponential expansion we have - after splitting even and odd terms -

$$e^{\theta K} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \theta^{2n} \mathbb{I} + \frac{(-1)^n}{(2n+1)!} \theta^{2n+1} K.$$

But these are just the expansions of the familiar triganometric functions,

$$e^{\theta K} = \cos\theta \mathbb{I} + \sin\theta K.$$

Which is just the rotation matrix[3] $R(\theta)$. □

[3]With a different sign convention than previously used.

Exercise 2. Repeat the above analysis for the case

$$Y = \sigma_+ + \sigma_-,$$

and show that Y generates the hyperbolic rotations $H(\theta)$, *i.e.* show that

$$H(\theta) = e^{\theta Y}.$$

Hence show that the matrix σ generates the dilatations, *i.e.* that

$$D(\lambda) = e^{\lambda\sigma}.$$

As a final example, let us consider the matrix generated by σ_+,

$$e^{x\sigma_+} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \sigma_+^n.$$

This series terminates almost immediately, as

$$\sigma_+^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence the series is just given by the upper triangular matrix

$$e^{x\sigma_+} = \mathbb{I} + x\sigma_+ = N(x).$$

Physically, this corresponds to a *Gallilean boost*, where

$$q \mapsto q + xp, \quad p \mapsto p.$$

One can think of this as a reference frame shift, as you might experience while seated in a car moving at constant velocity.

Exercise 3. Repeat this analysis with σ_-. Does this one-parameter subgroup also map to the additive group of the reals? What can you say about the transpose, in this case?

2 The Lie algebra of $\mathsf{Sp}(2n, \mathbb{R})$

Consider a *small* member of the symplectic group $\mathsf{Sp}(2n, \mathbb{R})$ continuously connect to the identity matrix, which we can parametrize by an infinitestimal ϵ,

$$M = \mathbb{I} + \epsilon\mu + \mathcal{O}(\epsilon)^2.$$

We can expand the symplectic condition

$$M^\mathsf{T} J M = J,$$

in powers of ϵ,

$$\left(\mathbb{I} + \epsilon\mu^\mathsf{T} + \mathcal{O}(\epsilon)^2\right) J(\mathbb{I} + \epsilon\mu + \mathcal{O}(\epsilon)^2) = J.$$

This equation is trivially true to zeroth order. To first order, we find the analogus defining relation for the Lie algebra $\mathfrak{sp}_{2n}\mathbb{R}$,

$$\mu^{\mathsf{T}} J + J\mu = 0. \tag{1}$$

That is to say, we have the group element defined in terms of μ by the exponential map

$$M(\epsilon) = e^{\epsilon\mu}.$$

Let us explore the consequences of (1).

Given that J is blocked into four $n \times n$ matrices, let us similarly block out μ,

$$\mu = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{2}$$

where A, B, C, D are $n \times n$ matrices. Plugging into (1),

$$\begin{pmatrix} A^{\mathsf{T}} & C^{\mathsf{T}} \\ B^{\mathsf{T}} & D^{\mathsf{T}} \end{pmatrix} \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

This implies

$$\begin{pmatrix} -C^{\mathsf{T}} & A^{\mathsf{T}} \\ -D^{\mathsf{T}} & B^{\mathsf{T}} \end{pmatrix} = \begin{pmatrix} -C & -D \\ A & B \end{pmatrix}.$$

Examining the blocks we find that originally off-diagonal blocks B and C are symmetric matrices,

$$B = B^{\mathsf{T}} \quad \text{and} \quad C = C^{\mathsf{T}}.$$

This implies that B and C each have $n(n+1)/2$ independent elements. The other two equatons imply the identical constraint[4],

$$A = -D^{\mathsf{T}}.$$

We can summarize this by writing

$$\left(\begin{array}{c|c} A & B \\ \hline C & -A^{\mathsf{T}}, \end{array}\right) \tag{3}$$

for B and C symmetric, and A just a square matrix.

This constraint halves the number of independent elements in the two sets of n^2 entries[5]. Hence, the total number of linearly independent generators in the symplectic group $\mathsf{Sp}(2n, \mathbb{R})$ - the number of basis elements of $\mathfrak{sp}_{2n}\mathbb{R}$ is

$$\dim \mathfrak{sp}_{2n}\mathbb{R} = 2n^2 + n.$$

We shall explore the physical ideas behind these individual generators in a later aside.

The representation theory of $\mathsf{Sp}(2n, \mathbb{R})$ has some interesting topological complications that are deeply related to the operator algebra in Quantum Mechanics. We shall begin our study of these representation next.

[4]Which for the case of $n = 1$ just amounts to the tracelessness condition

[5]Recall that the Lie algebra associated to $\mathsf{GL}(n, \mathbb{R})$ is essentially unconstrained, thanks to the exponential map construction.

3 Representative Examples of $\mathfrak{sp}(2n, \mathbb{R})$

3.1 qp-Rotations

It is amusing to note that while note a symplectomorphism, J is a member of the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$, as

$$J^\intercal = -J,$$

so

$$J^\intercal J + J^2 = 0.$$

Let us consider the group element generated by J,

$$e^{\alpha J} = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n J^n.$$

Notice that $J^2 = -\mathbb{I}$, and hence $J^3 = -J$, and $J^4 = \mathbb{I}$. This is precisely the same pattern we've seen with the rotation matrix generated by K in $\mathfrak{sl}(2, \mathbb{R})$, above. Hence,

$$e^\alpha J = \cos\alpha\mathbb{I} + \sin\alpha J.$$

Expanding this in block diagonal terms,

$$e^\alpha J = \left(\begin{array}{c|c} \cos\alpha\mathbb{I} & \sin\alpha\mathbb{I} \\ \hline -\sin\alpha\mathbb{I} & \cos\alpha\mathbb{I} \end{array}\right).$$

In terms of the phase space coordinates q^i and p_i, this transformation amounts to a *simulteneous* rotation amongst all the individual coordinate/momentum pairs,

$$q^i \mapsto \cos\alpha q^i + \sin\alpha p_i, \quad p_i \mapsto -\sin\alpha q^i + \cos\alpha p_i.$$

Exercise 4. How would you establish a pair-dependent rotation amongst all the q, p-pairs in phase space. That is, how would you generate a group element that rotates each pair (q^i, p_i) through a *distinct* angle α_i?

3.2 qq-Rotations

In general, a rotation matrix satisfies two conditions, first the det $R(\theta) = 1$, and second it is **orthogonal** in the sense that

$$R(\theta)^\intercal = R(\theta)^{-1}.$$

If we write our matrix in terms of some generator Q,

$$R(\theta) = e^{\theta Q},$$

then we have that[6]

$$R(\theta)^\intercal = e^{\theta Q^\intercal}.$$

But since $R(\theta)^\intercal$ equals $R(\theta)^{-1}$, we must have that $Q^\intercal = -Q$. That is, rotations are generated by antisymmetric matrices.

[6] To show this, it helps to observe that $(MN)^\intercal = N^\intercal M^\intercal$.

Now let Q be some matrix in the Lie algebra $\mathsf{sl}(2n, \mathbb{R}$. Using our block decomposition (3), suppose $B = C = 0$, and A is antisymmetric. Then

$$Q = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & -A^{\mathsf{T}} \end{array}\right).$$

But A is antisymmetric, so

$$Q = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right).$$

Since $R(\theta) = e^{\theta A}$ rotates amongst the coordinates q^i, we see that (as the solution to a previous exercise) this requires an identical, simultenous rotation amongst the momenta as well. That is to say,

$$M = e^{\theta Q} = \left(\begin{array}{c|c} R(\theta) & 0 \\ \hline 0 & R(\theta) \end{array}\right).$$

3.3 Nilpotent Matrices

A matrix M is said to be **nilpotent** if there exists some n such that $M^n = 0$. For example, the two-by-two matrices $\sigma_{\pm}$ are nilpotent. As a final example, let us consider a matrix Q in $\mathfrak{sp}(2n, \mathbb{R})$, where $A = C = 0$ in terms of (3). It is easy to see that this matrix is also nilpotent.

$$Q^2 = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right) = 0.$$

We can then readily expand the exponential,

$$e^{\alpha Q} = \left(\begin{array}{c|c} \mathbb{I} & \alpha B \\ \hline 0 & \mathbb{I} \end{array}\right).$$

This is another example of a "Gallilean boost", where we now send the phase space coordinates to

$$q^i \mapsto q^i + \alpha \sum_{j=1}^{n} B_{ij} p_j, \quad p_i \mapsto p_i.$$

Note that any nondiagonal B will impose boosts symmetrically on different coordinates.