

The Lie Group $\mathrm{Sp}(2n, \mathbb{R})$

Sean Downes

Episode #04 · October 28, 2025

The goal of this episode is to extend the description of the Lie group of linear symplectomorphisms to the case $\mathrm{Sp}(2n, \mathbb{R})$. To achieve that, we must first rewrite our model of phase space in a more symmetric way.

I Reframing Phase Space

Suppose $\dim \mathcal{P}$ is $2n$. We can then model it as the direct sum of vector spaces

$$\mathcal{P} = \mathbb{R}^n \oplus \mathbb{R}^n.$$

Practically, we may consider a basis formed by the q^i stacked above the p_i ,

$$\xi = \begin{pmatrix} \vec{q} \\ \vec{p} \end{pmatrix}.$$

In this case, the Hamilton equations can be written as

$$\dot{\xi} = -J \cdot \nabla_{\xi} E. \tag{1}$$

Here E is the same energy function, ∇_{ξ} is the gradient written in terms of our new consist basis, and J is given by the block matrix

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}.$$

Here J is said to be a **symplectic matrix**. To make sure you understand this idea and the notation, it helps to:

Exercise 1. Verify that (1) represents both of the Hamilton equations..

With this new notation, consider two real functions on $\mathcal{P}$, A and B . We can promote these two vectors in our new basis by taking their respective gradient,

$$\nabla_{\xi} A \quad \text{and} \quad \nabla_{\xi} B.$$

We can use the naïve inner product to create a bilinear map in the usual way,

$$\mathcal{F}(\mathcal{P}) \times \mathcal{F}(\mathcal{P}) \rightarrow \mathbb{R},$$

where

$$(A, B) \mapsto (\nabla_\xi A)^\top \nabla_\xi B,$$

but it's not clear what this means, mathematically or physically.

However, if we use the symplectic matrix J as a sort of “metric” for this inner product, we recover the Poisson Bracket.

$$\{A, B\} = (\nabla_\xi A)^\top \cdot J \cdot \nabla_\xi B, \quad (2)$$

Again, to make sure you understand the notation, you should verify this.

Exercise 2. Verify that the bracket in (2) is indeed the usual Poisson Bracket.

Remark 1.1. The previously mentioned fans of differential geometry will note that the symplectic two-form associated with the Poisson Bracket can be inferred directly from the definition of J ,

$$\omega = \sum_{i=1}^n dq^i \wedge dp_i.$$

1.1 The Symplectic Group

Now we are in a position to understand the general case of symplectomorphisms. Suppose M is a member of $\text{GL}(\mathcal{P})$, and let us act upon the gradient vectors $\nabla_\xi A$ in (2).

$$M : \{A, B\} \mapsto (M \cdot \nabla_\xi A)^\top \cdot JM \cdot \nabla_\xi B.$$

Given the familiar transpose relation $(MN)^\top = N^\top M^\top$, we see that

$$M : \{A, B\} \mapsto (\nabla_\xi A)^\top \cdot M^\top JM \cdot \nabla_\xi B.$$

Hence the Poisson Bracket is preserved by M if and only if

$$M^\top JM = J. \quad (3)$$

Matrices that preserve J in this fashion constitute the **Symplectic Group**, $\text{Sp}(2n, \mathbb{R})$, and constitute the linear symplectomorphism of $\mathcal{P}$. This group¹ belongs to the one of the four infinite families of classical Lie Groups, including the Special Orthogonal and Special Unitary groups.

Proposition 1.2. The Lie group $\text{Sp}(2n, \mathbb{R})$ is a proper subgroup of $\text{GL}(2n, \mathbb{R})$.

Proof. From (3), we see

$$\det(MJM) = \det J \Rightarrow M = \pm 1.$$

□

In fact, we can show more.

¹Or rather, the analog over the complex numbers.

Proposition 1.3. The Lie group $\mathrm{Sp}(2n, \mathbb{R})$ is a subgroup of $\mathrm{SL}(2n, \mathbb{R})$, and proper for $n > 1$.

Proof. To show that $\det M$ in fact equals 1, one can observe that J is an even-dimensional antisymmetric matrix, and take the *Pfaffian* of both sides. In this case there is a known relation²,

$$\mathrm{Pf}(M^T J M) = \det M \mathrm{Pf}(J).$$

Hence $\det M = 1$.

We have already seen that $\mathrm{Sp}(2, \mathbb{R})$ equals $\mathrm{SL}(2, \mathbb{R})$. To show that $\mathrm{Sp}(2n, \mathbb{R})$ is a proper subgroup of $\mathrm{SL}(2n, \mathbb{R})$ for $n > 1$, we need only find an element in $\mathrm{SL}(2n, \mathbb{R})$ that is not in $\mathrm{Sp}(2n, \mathbb{R})$.

To find such an element, let x and y be two coordinates with associated momenta, p_x and p_y . Now consider a matrix M that rotates x into y but leaves p_x and p_y alone. For $n = 2$ this matrix can be written,

$$M = \left(\begin{array}{c|c} R(\theta) & 0 \\ \hline 0 & \mathbb{I} \end{array} \right),$$

where $R(\theta)$ is the usual rotation matrix in two-dimensions.

We can then compute,

$$M^T J M = \left(\begin{array}{c|c} R(\theta)^T & 0 \\ \hline 0 & \mathbb{I} \end{array} \right) \left(\begin{array}{c|c} 0 & -\mathbb{I} \\ \hline \mathbb{I} & 0 \end{array} \right) \left(\begin{array}{c|c} R(\theta) & 0 \\ \hline 0 & \mathbb{I} \end{array} \right) = \left(\begin{array}{c|c} R(\theta)^T & 0 \\ \hline 0 & \mathbb{I} \end{array} \right) \left(\begin{array}{c|c} 0 & -\mathbb{I} \\ \hline R(\theta) & 0 \end{array} \right).$$

Hence

$$M^T J M = \left(\begin{array}{c|c} 0 & R(\theta)^T \\ \hline R(\theta) & 0 \end{array} \right) \neq J.$$

Hence this M , while in $\mathrm{SL}(2n, \mathbb{R})$, it is not in $\mathrm{Sp}(2n, \mathbb{R})$. □

Exercise 3. Find a member of $\mathrm{Sp}(2n, \mathbb{R})$ that rotates x into y as in the previous example, but also transforms the momenta as needed.

²As with everything related to the Pfaffian, this proof immediately breaks into an exercise in combinatorics. But it helps to observe that the determinant is a maximally antisymmetric product of components of M .