The Pasayten Institute

The Lie Group SL2R

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The goal of this episode is to identity and study the Lie group of linear transformations of two-dimensional phase space the preserve the Poisson Bracket. We will identity the associated Lie algebra and a concrete construction of it.

1 Linear Transformations and the Poisson Bracket

Given a finite-dimensional vector space V, any two nonzero members v, w of V can be related by a linear transformation, M,

$$w = Mv, \quad v = M^{-1}w.$$

Put differently, we say that the general linear group $\mathsf{GL}(V)$ acts transitively on the nonzero vectors V. You can show this by explicitly producing such a matrix once you specify a basis.

The set of all such invertible matrices constitutes the group of *automorphisms* of V. By definition, these are the maps that preserve its structure as a vector space. In this case, the most important features are *linearity* and the *dimension*.

Let $\mathcal{P}$ be the $2n$-dimensional phase space associated to n independent position vectors q^i and their associated momentum vectors p_i. This vector space - which we use to study the dynamics of physical objects - comes equipped with additional structure. Any rotation of the position vectors should have a corresponding rotation amongst the momenta. It is sometimes said that p_i is *soldered* to q^i. This is the meaning of the second Hamilton equation,

$$\dot{q}^i = \frac{\partial E}{\partial p_i}.$$

An automorphism of $\mathcal{P}$ that preserves the Hamilton equations preserves the physics. For historical reasons such an automorphism has been called a **symplectomorphism**[1]

Given our previous discussion, it's also clear that such a symplectomorphism of phase space induces an isomorphism of the Lie algebra $(\mathcal{F}(\mathcal{P}), \{\cdot, \cdot\})$. Indeed, this fact offers a simple way to detect which members of $\mathsf{GL}(\mathcal{P})$ are symplectomorphisms. A symplectomorphism should preserve the Poisson Bracket.

[1]The notion of a symplectomorphism extends to the general case where $\mathcal{P}$ is a manifold. In the present discussion we shall restrict ourselves to the case of *linear* symplectomorphisms, as the general case - even or perhaps especially on $\mathbb{R}^4$ - is wildly complicated.

2 The Case of Two Dimensions

For a single pair q, p, the most general linear transformation is a matrix,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

M acts on $\mathcal{P}$ in such a way that,

$$M : q \mapsto aq + bp, \quad M : p \mapsto cq + dp.$$

The Poisson bracket of these transformed quantities is,

$$\{aq + bp, cq + dp\} = (ad - bc)\{q, p\} = \det M$$

.

Immediately we see that M must have unit determinant. In this case one can show that for any two functions on phase space A and B,

$$M : \{A, B\} \mapsto \frac{1}{\det M}\{A, B\}.$$

And so we see that a necessary and sufficent condition for M to preserve the Poisson Bracket in two-dimensions is that M belong to the special linear group, $\mathsf{SL}(2, \mathbb{R})$. This is precisely the subgroup of $\mathsf{GL}(2, \mathbb{R})$ whose elements have unit determinant.

Proposition 2.1. $\mathsf{SL}(2, \mathbb{R})$ is a proper subgroup of $\mathsf{GL}(2, \mathbb{R})$.

Proof. Let M and N be two matrices in $\mathsf{SL}(2, \mathbb{R})$ and consider their product MN. By rules of determinants,

$$\det(MN) = \det M \det N.$$

By hypothesis, $\det M = \det N = 1$, hence $\det MN$ is also 1, and hence MN belongs to $\mathsf{SL}(2, \mathbb{R})$.

The identity matrix also has unit determinant. For each M, the inverse matrix M^{-1} must satisfy

$$MM^{-1} = \mathbb{I}.$$

Hence by the above argument, $\det M^{-1}$ is also 1. Hence $\mathsf{SL}(2, \mathbb{R})$ is a subgroup of $\mathsf{GL}(2, \mathbb{R})$.

To show that it is a proper subgroup, we need only note that the determinant of a matrix in $\mathsf{GL}(2, \mathbb{R})$ may take any nonzero value, which in particular may differ from 1. Hence there are plenty of matrices in $\mathsf{GL}(2, \mathbb{R})$ that are not in $\mathsf{SL}(2, \mathbb{R})$.

□

Proposition 2.2. The dimension of $\mathsf{SL}(2, \mathbb{R})$ is three.

Proof. The dimension of any two-by-two matrix is at most four, since there are only four entries to parametrize the group. Indeed, $\dim \mathsf{GL}(2, \mathbb{R})$ is four. For $\mathsf{SL}(2, \mathbb{R})$, this dimensionality is reduced by the constraint that for any M defined as above,

$$\det M = ad - bc = 1.$$

As the parameters a, b, c, d are otherwise unobstructed, this reduces the dimensionality of $\mathsf{SL}(2, \mathbb{R})$ to three.

□

3 Examples

3.1 Rotations

As a first example, consider the rotation matrix $R(\theta)$,

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

We can compute,

$$\det R(\theta) = \cos^2\theta - (-\sin^2\theta) = \cos^2\theta + \sin^2\theta = 1.$$

Hence $R(\theta)$ belongs to $\mathsf{SL}(2,\mathbb{R})$.

This matrix transforms the members of $\mathfrak{P}$ as follows,

$$q \mapsto \cos\theta q + \sin\theta p, \quad p \mapsto -\sin\theta q + \cos\theta p.$$

Why you would *want* to rotate q and p is another question, which we will get to in due course.

3.2 Hyperbolic Rotations

Next consider

$$H(\theta) = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix}.$$

We can then compute,

$$\det H(\theta) = \cosh^2\theta - \sinh^2\theta = 1.$$

Hence again, $H(\theta)$ belongs to $\mathsf{SL}(2,\mathbb{R})$.

This leads to a pretty wild transformation of the coordinates,

$$q \mapsto \cosh\theta q + \sinh\theta p, \quad p \mapsto \sinh\theta q + \cosh\theta p.$$

From a physicist's perspective, this corresponds to a Lorentz Boost familiar from Special Relativity.

3.3 Dilatations

As a third example, we can consider

$$D(\lambda) = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{pmatrix}.$$

We can compute,

$$\det D(\lambda) = e^\lambda e^{-\lambda} = 1,$$

Hence $D(\lambda)$ belongs to $\mathsf{SL}(2,\mathbb{R})$.

This leads to a pretty gnarly distortion of phase space,

$$q \mapsto e^\lambda q, \quad p \mapsto e^{-\lambda} p.$$

In a future example, we will see how to employ this mapping in a practical way.

3.4 A Nilpotent Case

Consider the following "parabolic" element,

$$N(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

This matrix clearly has determinant 1. What's interesting about this matrix is it's algebra,

Proposition 3.1. The upper triangular matrices $N(x)$ form a proper subgroup of $\mathsf{SL}(2,\mathbb{R})$ isomorphic to the additive group of the reals, $(\mathbb{R},+)$.

Proof. Consider the product of two elements $N(x)$ and $N(y)$.

$$N(x)N(y) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+y \\ 0 & 1 \end{pmatrix}.$$

The identity matrix $\mathbb{I}$ belongs to this class of matrices, $N(0)$. By the above calculation, we see that $N(x)N(-x)$ equals $N(0)$. Hence every such matrix has an inverse. So it is a group, and a proper subgroup at that.

The isomorphism between this subgroup and $\mathbb{R}$ is explicit.

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