Time Evolution and the Adjoint Operator

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The aim of this lecture is give a concrete demonstration of the exponential map between the Lie algebra and its associated Lie group. Specifically, we'll use the Lie algebra $\mathcal{F}(\mathcal{P})$ and the energy function to study the time evolution of a physical system.

1 The Adjoint Operator

From the Hamilton equations we have already seen that the time derivative of a function over $\mathcal{P}$ is given by,

$$\frac{dA}{dt} = \{A, E\}.$$

Let us take the Leibniz notation for the time derivative seriously and consider it as an operator on our Lie algebra $\mathcal{F}(\mathcal{P})$. To bring in a little Lie theory, the **adjoint operator** associated to an element B of $\mathcal{F}(\mathcal{P})$ is a linear endomorphism[1] ad_B:

$$\mathsf{ad}_B : \mathcal{F}(\mathcal{P}) \to \mathcal{F}(\mathcal{P}),$$

defined by its action on some element A,

$$\mathsf{ad}_B \cdot A = \{A, B\}.$$

The idea is that the adjoint operator build from B means "take the Lie bracket with B". Hence, we may explicitly represent the time derivative as an adjoint operator,

$$\frac{d}{dt} \rightsquigarrow \mathsf{ad}_E. \tag{1}$$

Incidentally, the adjoint operators form a representation of $\mathcal{F}(\mathcal{P})$ under the commutator of composition maps,

$$\{A, B\} \leftrightsquigarrow [\mathsf{ad}_A, \mathsf{ad}_B] = \mathsf{ad}_A \circ \mathsf{ad}_B - \mathsf{ad}_B \circ \mathsf{ad}_A.$$

By this we mean that if $C = \{A, B\}$, then

$$[\mathsf{ad}_A, \mathsf{ad}_B] = \mathsf{ad}_C,$$

which the pendants can verify directly as an exercise.

[1] That is, a map from a given space to itself.

2 The Exponential Map

One way to understand the relationship between Lie groups and their Lie algebras is to think about analytic functions. Given (??), we know that $q^i(t)$ and $p_i(t)$ are functions of time and that hence any $A(q,p)$ in $\mathcal{F}(\mathcal{P})$ is also an *implicit* function of time. Assuming that A is analytic at some point in phase space, this means that we should have a Taylor expansion,

$$A(t^\star) = \sum_{n=0}^{\infty} \frac{1}{n!}(t^\star - t)^n \frac{d^n}{dt^n} A.$$

Here we have chosen $t^\star$ to be the specific point in time corresponding to a specific point in $\mathcal{P}$ on a trajectory at a fixed energy E. What's most important for our purposes is the presence of the an infinite series of time derivatives. Let us express this in a more familiar way. Let α be a real number. Then,

$$\exp(\alpha \frac{d}{dt}) = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n \frac{d^n}{dt^n}.$$

If we think of α as the amount of time traversed, $t^\star - t$, then this operator effectively generates a time translation of some function $A(t)$ by way of the Taylor expansion,

$$\exp(\alpha \frac{d}{dt}) \cdot A(t) = A(t + \alpha).$$

Using our identification (1), we have a representation of the time evolution on $\mathcal{F}(P)$ in terms of the Poisson bracket.

$$e^{\alpha \mathsf{ad}_E} \cdot A(q(t), p(t)) = A(q(t+\alpha), p(t+\alpha)).$$

To recapitulate: The expontential map from the Lie algebra produces a member of the Lie group whose actions on the function $A(t)$ hinges on the Taylor series representation of an analytic function.

3 The Time Evolution of Position

3.1 The Time Evolution of Position

Let us consider a basic and important example. Suppose there is one degree of freedom with coordinate q, and consider the action of

$$e^{\alpha \mathsf{ad}_E} \cdot q = \sum_{n=0}^{\infty} \frac{1}{n!} \alpha^n (\mathsf{ad}_E^n \cdot q).$$

To explicitly represent this summation we need to compute some Poisson brackets. The first couple are straightforward,

$$\{q, E\} = \frac{p}{m}, \quad \{\{q, E\}, E\} = \frac{1}{m}\{p, E\} = -\frac{1}{m}\frac{dV}{dq}.$$

After the third,

$$\mathsf{ad}_E^3 \cdot q = -\frac{1}{m}\{V'(q), E\} = -\frac{1}{m^2} V''(q) p,$$

we see that things start to get convoluted,

$$\mathsf{ad}_E^4 \cdot q = -\frac{1}{m^2}\left(\frac{1}{m}V'''(q)p^2 - V''(q)V'(q)\right).$$

You are welcome to continue this series for another term or two. It's pretty instructive for how these complications arise.

Of course, this all makes sense. More complicated potentials can lead to more complicated dynamics. The main culprit that causes multiplicative mixing of p and q appears to be the q dependence of $V''(q)$. Let's restrict our attention to a couple of specific cases before these complication arise.

3.2 A Constant Potential

If $V = V_0$, then our series terminates at ad_E^2,

$$\mathsf{ad}_E \cdot q = \frac{p}{m} \quad \mathsf{ad}_E^2 \cdot q = 0.$$

Hence we have the expansion

$$e^{\alpha \mathsf{ad}_E} \cdot q = q + \alpha\{q, E\} = q + \frac{p}{m}\alpha.$$

We know from (**??**) that such a case implies that $\dot{p} = 0$, and hence the object has merely moved with constant velocity.

$$q(t+\alpha) = q(t) + \frac{p}{m}\alpha.$$

If you like, we can be more explicit. From the definition of E,

$$p = \pm\sqrt{2m(E - V_0)}.$$

The sign of course represents the direction of motion. Hence if q_0 is the position of the object at time $t = 0$,

$$q(t) = q_0 \pm \sqrt{\frac{2m(E-V_0)}{m}}t,$$

and so

$$e^{\alpha \mathsf{ad}_E} \cdot q = q_0 \pm \sqrt{\frac{2m(E-V_0)}{m}}(t+\alpha).$$

3.3 A Linear Potential

If $V(q) = V_0 + aq$, with a a constant, the series terminates at ad_E^3,

$$\mathsf{ad}_E \cdot q = \frac{p}{m}, \quad \mathsf{ad}_E^2 \cdot q = -\frac{a}{m}, \quad \mathsf{ad}_E^3 \cdot q = 0.$$

In this case,

$$e^{\alpha \mathsf{ad}_E} \cdot q = q + \frac{p}{m}\alpha - \frac{a}{2m}\alpha^2.$$

This is constant accelerated motion, familiar from the study of the force of Gravity on the Earth's surface. Solving the Hamilton equations, we find that

$$q(t) = -\frac{a}{2m}t^2 + \beta t + \gamma,$$

for two integration constants β and γ. Hence

$$p(t) = -\frac{a}{m}t + \beta.$$

Plugging this in above we find,

$$e^{\alpha \mathsf{ad}_E} \cdot q = -\frac{a}{2m}(t+\alpha)^2 + \beta(t+\alpha) + \gamma.$$

3.4 A Quadratic Potential

Things get interesting once V becomes quadratic in q. While we certainly can consider a linear term in our potential, for simplicitly, let shift our definition of q to center at the local minimum:

$$V = V_0 + \frac{1}{2}kq^2.$$

Note that any such shift generically changes the value of V_0.

In any case, we see that the series of adjoint operators no longer terminates, but rather *alternates*,

$$\mathsf{ad}_E \cdot q = \frac{p}{m}, \quad \mathsf{ad}_E^2 \cdot q = -\frac{k}{m}q.$$

Iterating,

$$\mathsf{ad}_E^3 \cdot q = -\left(\frac{k}{m}\right)\frac{p}{m}, \quad \mathsf{ad}_E^4 \cdot q = \left(\frac{k}{m}\right)^2 q.$$

and hence

$$\mathsf{ad}_E^{2n+1} \cdot q = \left(-\frac{k}{m}\right)^n \frac{p}{m}, \quad \mathsf{ad}_E^{2n} \cdot q = \left(-\frac{k}{m}\right)^{2n} q,$$

which you can readily verify by induction.

The main point is that even powers of ad_E result in a term proportional to q, and odd powers result in a term proportional to p/m. Hence the series expansion of the exponential breaks down in a familiar way:

$$e^{\alpha \mathsf{ad}_E} \cdot q = \cos(\omega\alpha)q + \sin(\omega\alpha)\frac{p}{m}, \tag{2}$$

where

$$\omega = \pm\sqrt{\frac{k}{m}}.$$

Again the choice of sign amounts to the direction of motion. Now, solving the equations of motion, we know that for a suitable choice of initial position,

$$q(t) = a\cos(\omega t),$$

where the amplitude a is related to the total energy of the object. Hence,

$$p(t) = -am\omega \sin(\omega t).$$

Plugging into (2),

$$e^{\alpha \mathsf{ad}_E} \cdot q(t) = a\cos(\omega\alpha)\cos(\omega t) - a\sin(\omega\alpha)\sin(\omega t).$$

Using the familiar triganometric identity,

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

we see that this amounts to

$$e^{\alpha \mathsf{ad}_E} \cdot q(t) = a\cos(\omega(t+\alpha)).$$

As we've seen above, as soon as the potential becomes cubic or higher, the resulting equations of motion are nonlinear and the associated Poisson brackets become convoluted. You are welcome to explore higher powers on your own to get a feel for how out of hand things can get. There is plenty of opportunity to explore applications of number theory, including restricted partitions of integers.