

The Poisson Bracket is a Lie bracket

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Today we are going to describe the infinite-dimensional Lie algebra of real, analytic functions on the phase space of a dynamical system.

I Classical Dynamics

I.1 Conservative Forces

Let's begin with the basics. Consider an object of inertial mass m moving nonrelativistically under the influence of a conservative force¹, $\vec{F}$. Let $\vec{q}$ parametrize its position in n -dimensional space.

A **conservative force** is a force that may be written as the gradient of a scalar function, typically called the **potential** or **potential energy**

$$\vec{F} = -\vec{\nabla}V(\vec{q}). \quad (1)$$

If $\vec{F}$ is the only force relevant for our discussion, we can the system **conservative**. We will show that the energy is constant in time². The **energy** is defined as the sum of the kinetic and potential energies

$$E = \frac{\vec{p} \cdot \vec{p}}{2m} + V(\vec{q}). \quad (2)$$

Here we define $\vec{p}$ to be the **momentum**³ of the object

$$\vec{p} = m \frac{d\vec{q}}{dt}. \quad (3)$$

In some sense, conservative systems are described by the exchange of kinetic and potential energies.

¹These technical details are why we assume a basic background in elementary physics. Nonrelativistic means that the speed of the object is very small compared with the speed of light, $c \approx 3 \times 10^8$ m/s. Forces push on objects via Newton's Second Law, which states that the vector sum of all forces is directly proportional to the acceleration of an object. The inertial mass is the constant of proportionality. We define the notion of a conservative force immediately below.

²Hence the name, *conservative force*. Nonconservative forces cannot be written in this form, and hence the energy must be defined carefully in terms of the *Work-Energy Theorem*. This adds an explicit time-dependence to everything we will be discussing below.

³Implicit in this definition is the assumption that the object itself is indivisible. Momentum can be different depending upon the details of the physical system, although there is typically a canonical way to define it given said system.

Proposition 1.1. The energy of a conservative system is constant in time.

Proof. Consider the time derivative of the energy, $\dot{E}$:

$$\dot{E} = \frac{1}{m} \vec{p} \cdot \frac{d\vec{p}}{dt} + \vec{\nabla} V \cdot \frac{d\vec{q}}{dt}.$$

Applying the definition of the momentum we see $\dot{E}$ factorizes,

$$\dot{E} = \frac{1}{m} \vec{p} \cdot \left(\frac{d\vec{p}}{dt} + \vec{\nabla} V(\vec{q}) \right).$$

Given that the inertial mass is constant, we see that the time derivative of the momentum is related to the acceleration,

$$\frac{d\vec{p}}{dt} = m \frac{d^2\vec{q}}{dt^2}. \quad (4)$$

Because there is only one force under consideration, Newton's Second Law of motion reads,

$$\vec{F} = m \frac{d^2\vec{q}}{dt^2},$$

But the force is given by (1). Hence $\dot{E}$ vanishes⁴ □

1.2 The Hamilton Equations

For conservative systems, the state of a physical system is determined by the energy. Newton's Second Law - (4) - amounts to a second order differential equation for the position of the object as a function of time, $\vec{q}(t)$. For reasons that will become clear, it is traditional to reframe this as a system of two, *first order* differential equations.

Proposition 1.2. For a conservative system, the equations of motion - (4) - can be expressed as

$$\dot{p}_i = -\frac{\partial E}{\partial q^i}, \quad \dot{q}^i = \frac{\partial E}{\partial p_i}. \quad (5)$$

Proof. The energy defined in (2) is quadratic form in momentum. Hence, for individual vector components,

$$\frac{\partial E}{\partial p_i} = \frac{1}{m} p_i.$$

Given the definition of momentum, (3), we see that this is just $\dot{q}^i$.

The explicit coordinate dependence of E enters only through the potential, so

$$\frac{\partial E}{\partial q^i} = \frac{\partial V}{\partial q^i},$$

which is the definition of a force. We have already seen, in particular, that this is just $\dot{p}_i$.

Putting these two results together, we recover (4). □

⁴In the video we worked this the other way around. This is for two reasons. First, it was faster. Second, it demonstrates (by implicit contradiction referenced in the video footnote) that we cannot just assume that the energy is constant. It's a consequence of Newton's Second Law. I hope that narrative device wasn't too confusing.

Exercise 1. Generalize E in (2) to consider N independent objects, each with their own distinct inertial mass m_i . Hence verify that (5) remain the same despite this choice.

Exercise 2. Consider the same system in spherical polar coordinates,

$$x = r \cos \theta \sin \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta.$$

Write the explicit form of $q_1 = r$, $q_2 = \theta$ and $q_3 = \phi$ in terms of the rectilinear coordinates x, y and z . Compute their associated momenta. Hence verify that the Hamilton equations remain unchanged. For simplicity, you may assume $V = V(r)$.

2 Phase Space

The Hamilton equations describe the dynamics of a physical system in terms of two sets of related variables, position and momenta. Given that energy is constant in time, we can use it to parametrize curves in a $2n$ -dimensional vector space⁵ spanned by the n components of the position vector q^i and their associated momenta p_i . Such curves are called **orbits** or **trajectories**.

This $2n$ -dimensional stage on which the dynamics play out is called **Phase Space**, and is generically represented by $\mathcal{P}$. We can typically think of it as the direct sum of vector spaces of position and momenta,

$$\mathcal{P} = \mathbb{R}^n \oplus \mathbb{R}^n.$$

Observables in classical dynamics are given by real functions over $\mathcal{P}$. Let A be such a function,

$$A : \mathcal{P} \rightarrow \mathbb{R}.$$

In order to make predictions for what we observe, it is important to know how observables like A behave as a function of time. The time derivative of A can be expressed via the chain rule,

$$\dot{A} = \sum_{i=1}^n \frac{\partial A}{\partial q^i} \dot{q}^i + \frac{\partial A}{\partial p_i} \dot{p}_i.$$

Using the Hamilton equations, this becomes

$$\dot{A} = \sum_{i=1}^n \frac{\partial A}{\partial q^i} \frac{\partial E}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial E}{\partial q^i}. \quad (6)$$

Antisymmetrized derivatives are a common sight in differential geometry, and a trained mathematician will know exactly what to do with (6). The rest of us will need to take a leap of faith while we generalize.

2.1 The Poisson Bracket

Let A and B be any two analytic functions over phase space and consider their **Poisson Bracket**

$$\{A, B\} = \sum_{i=1}^n \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q^i}. \quad (7)$$

⁵We are implicitly assuming that $\vec{q}$ is a vector - or at least a coordinate from a given chart on a vector space. These ideas can generalize to manifolds with nonzero curvature, but not without considerable complication in what follows.

Immediately we see that for any such function A ,

$$\dot{A} = \{A, E\}.$$

In particular, the Hamilton equations become

$$\dot{q}^i = \{q^i, E\}, \quad \dot{p}_i = \{p_i, E\}. \quad (8)$$

But there's more to this bracket than just time evolution.

Exercise 3. Verify explicitly that (8) imply the Hamilton equations. Show also that

$$\{q^i, p_j\} = \delta_j^i, \quad \{q^i, q^j\} = \{p_i, p_j\} = 0.$$

Proposition 2.1. The Poisson Bracket is antisymmetric and bilinear in its arguments. It satisfies the **Leibniz condition**, where for any real analytic functions A, B, C ,

$$\{AB, C\} = \{A, C\}B + A\{B, C\}.$$

It also satisfies the Jacobi identity,

$$\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0.$$

Proof. Antisymmetry is clear from the definition (7). Linearity in each argument follows from the fact that partial derivative is a linear operator,

$$\frac{d}{dx}(af(x) + bg(x)) = af'(x) + bg'(x).$$

The Leibniz condition follows directly from the product rule,

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Finally, the Jacobi identity we leave to the reader as an exercise⁶. □

This proposition is often taken as the formal definition of a Poisson bracket. In such a case, we have

Corollary 2.2. A Poisson Bracket is a Lie Bracket.

Proof. A Lie bracket is a bilinear map on a vector space that is alternating⁷ and satisfies the Jacobi identity. Antisymmetry and the Jacobi identity together imply the bracket is alternating. Consider the Jacobi identity when $C = A$,

$$\{\{A, B\}, A\} + \{\{B, A\}, A\} + \{\{A, A\}, B\} = 0.$$

Antisymmetry cancels the first two terms. Since this must hold for any A and B , we see that $\{A, A\}$ vanishes. Hence the Poisson Bracket is a Lie bracket. □

We therefore see that the vector space of appropriate⁸ functions over $\mathcal{P}$ form a Lie algebra under the Poisson Bracket.

⁶We all have to do it once. You might as well start now.

⁷A bracket is **alternating** if $\{A, A\}$ vanishes for all A .

⁸You can substitute analytic here if desired, although more general cases can be made to hold. If it's not immediately obvious to you why a given collection of functions should be a vector space, consider it an exercise to explore the definition and explain to yourself why this is true.